7.4

Partial Fractions and Partial Fraction Decomposition

Given a fraction like $\frac{5x-1}{x^2+x-12}$ *can be written as* $\frac{3}{x+4}+\frac{2}{x-3}$ this is referred to as partial fraction decomposition and the two simpler fractions are called partial fractions.

$\frac{P}{Q}$ is called proper if the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator. Otherwise, the rational expression is termed improper.

The partial fraction decomposition of the rational expression $\frac{P}{Q}$ depends on the factors of the denominator Q. The denominator Q will contain only factors of one or both of the following types:

Linear factors of the form *x* - *a*, where *a* is a real number

Irreducible quadratic factors of the form, where *a, b*, and *c* are real numbers and $a \neq 0$ and $b^2 - 4ac < 0$.

There are four cases to be examined:

Case 1: Q has only nonrepeated linear factors.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \ldots + \frac{A_n}{x-a_n}$$

Case 2: Q has repeated linear factors.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_n}{(x-a)^n}$$

Case 3: Q contains a nonrepeated irreducible quadratic factor

$$\frac{P(x)}{Q(x)} = \frac{Ax+B}{ax^2+bx+c}$$

Case 4: Q contains repeated irreducible quadratic factors

$$\frac{P(x)}{Q(x)} = \frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \ldots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

Ex. Case 1

Find the partial fraction decomposition of

$$\frac{5x-1}{x^2-2x-15}.$$

The first step is to factor the denominator.

$x^2-2x-15=(x-5)(x+3)$

Using the form in case 1 $\frac{5x-1}{x^2-2x-15}$ = $\frac{A_1}{x-5}+\frac{A_2}{x+3}$

Multiplying both sides of the equation by $x^2-2x-15$ we get:

$$5x-1=A_1(x+3)+A_2(x-5)$$

$$5x-1=A_1x+3A_1+A_2x-5A_2$$

So,

$$A_1+A_2=5$$
$$3A_1-5A_2=-1$$

We can write this system in matrix form as BX = C where

B = $\begin{bmatrix}1 & 1\\3 & -5\end{bmatrix}, \quad X=\begin{bmatrix}A_1\\A_2\end{bmatrix} \quad and \quad C=\begin{bmatrix}5\\-1\end{bmatrix}$

$[B]^{-1}[C]=[X] \quad [X]=\begin{bmatrix}3\\2\end{bmatrix}$

So, $A_1=3 \ \ and \ \ A_2=2$ therefore,

$\frac{5x-1}{x^2-2x-15} = \frac{3}{x-5}+\frac{2}{x+3}$.

Ex. Case 2

Find the partial fraction decomposition of

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}.$$

Factoring the denominator we get $x(x-2)^2$. Because the factor (x – 2) is squared, it contributes two terms to the decomposition:

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}=\frac{A_1}{x}+\frac{A_2}{x-2}+\frac{A_3}{\left(x-2\right)^2}$$

Clearing fractions by multiplying both sides of the equation by x^3-4x^2+4x gives:

$$-x^2+2x+4=A_1\left(x-2\right)^2+A_2x\left(x-2\right)+A_3x$$

Expanding and combining like terms in the above equation we get:

$$-x^2+2x+4=(A_1+A_2)x^2+\left(-4A_1-2A_2+A_3\right)x+4A_1$$

which gives the following system of equations:

$$\begin{aligned}&A_1+A_2=-1\\&-4A_1-2A_2+A_3=2\\&4A_1=4\end{aligned}$$

Using matrices: $\begin{bmatrix} 1 & 1 & 0 & -1 \\ -4 & -2 & 1 & 2 \\ 4 & 0 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

So, $A_1 = 1,\ A_2 = -2,\ A_3 = 2$

and $$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{1}{x} + \frac{-2}{x-2} + \frac{2}{(x-2)^2}.$$

Ex. Case 3

Find the partial fraction decomposition of:

$$\frac{x^2+4x+1}{x^3-x^2+x-1}.$$

Factoring the denominator by grouping we get:

$$x^3-x^2+x-1 = x^2(x-1)+(x-1) = (x-1)(x^2+1)$$

Since each factor occurs once, each one leads to one term in the decomposition:

$$\frac{x^2+4x+1}{x^3-x^2+x-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Multiplying both sides of the equation by the denominator to clear fractions we get:

$$x^2+4x+1 = A(x^2+1)+(Bx+C)(x-1).$$

Expanding and combining like terms gives:

$$x^2+4x+1 = (A+B)x^2+(-B+C)x+(A-C).$$

We then obtain the following system by comparing coefficients on the left and right sides of the equation:

$$\begin{aligned} A+B&=1 \\ -B+C&=4 \\ A-C&=1 \end{aligned}$$

From here we can use matrices to find that A = 3, B = -2, and C = 2.

So, $\dfrac{x^2+4x+1}{x^3-x^2+x-1}=\dfrac{3}{x-1}+\dfrac{-2x+2}{x^2+1}$.

Ex. Case 4

Find the partial fraction decomposition of:

$$\frac{2x^3-x^2+5x}{(x^2+1)^2}.$$

The factor $(x^2+1)^2$ in the denominator leads to two terms in the partial fraction decomposition:

$$\frac{2x^3-x^2+5x}{(x^2+1)^2}=\frac{B_1x+C_1}{x^2+1}+\frac{B_2x+C_2}{(x^2+1)^2}.$$

Clearing fractions by multiplying both sides of the equation by $(x^2+1)^2$ gives:

$$\begin{aligned} 2x^3-x^2+5x&=(B_1x+C_1)(x^2+1)+B_2x+C_2 \\ &=B_1x^3+C_1x^2+(B_1+B_2)x+(C_1+C_2). \end{aligned}$$

Comparing coefficients of x on the left and right sides of the equation we can see that $B_1=2, C_1=-1, B_1+B_2=5, and\ C_1+C_2=0$. So, $B_2=3, C_2=1$ therefore we get:

$$\frac{2x^3-x^2+5x}{(x^2+1)^2}=\frac{2x-1}{x^2+1}+\frac{3x+1}{(x^2+1)^2}.$$